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Punctured torus groups and two-parabolic groups

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This paper is a progress report of the research aimed at the following objectives:

(1) To get precise understanding of Jorgensen's unfinished work [3] on the space of quasi-fuchsian groups representing a pair of punctured tori and to understand his construction of the hyperbolic structures of punctured torus bundles over a circle.

(2) To establish an analogue of Jorgensen's work for the Riley slice of Schottky space (i.e., the space of discrete free subgroups of $\mathrm{PSL}(2, \boldsymbol{C})$ generated by two parabolic transformations).

(3) To understand the hyperbolic structures of the 2-bridge knot complements (equivalently, the 3-dimensional hyperbolic manifolds of finite volume whose fundamental groups are generated by two parabolic transformations) from the above view point, and to prove the conjecture proposed by the second author and J. Weeks [9] on the combinatorial structure of their Ford domains.

In this paper, we describe the following:

(1) Our interpretation of Jorgensen's work.

(2) An analogue of Jorgensen's result for the Riley slice of Schottky groups.

(3) A certain conjecture on the existence and the shape of continuous families of hyperbolic cone manifolds joining "rational" boundary points of the Riley slice and 2-bridge knot complements: an affirmative answer to the conjecture implies that to the conjecture in [9].

Unfortunately, we have not been able to prove the whole of the assertions in (1) and (2) as yet. The continuous families of cone manifolds in (3) can be considered as continuations of the rational pleating rays discussed by Keen-Series [4] and Komori-Series [5]. We present some computer experiments, based on a software written by the third author [11] which support the conjecture. In particular, we construct the hyperbolic structure of the figure-eight knot complement from this view point.

1. Fricke surfaces and the modular diagram

Let T, S, $\mathcal{O}$, respectively, be a (once) punctured torus, a 4-times punctured sphere, and a $(2,2,2,\infty)$-orbifold (i.e., the orbifold with underlying space a punctured sphere and with three cone points of index 2). They have $\boldsymbol{R}^2 - \boldsymbol{Z}^2$ as the common covering space. To be precise, let Γ and $\tilde{\Gamma}$, respectively, be the groups of transformations on $\boldsymbol{R}^2 - \boldsymbol{Z}^2$ generated by π-rotations about points in $\boldsymbol{Z}^2$ and $(\frac{1}{2}\boldsymbol{Z})^2$. Then $T = (\boldsymbol{R}^2 - \boldsymbol{Z}^2)/\boldsymbol{Z}^2$, $S = (\boldsymbol{R}^2 - \boldsymbol{Z}^2)/\Gamma$ and $\mathcal{O} = (\boldsymbol{R}^2 - \boldsymbol{Z}^2)/\tilde{\Gamma}$. In particular, there is a $\boldsymbol{Z}_2$-covering $T \to \mathcal{O}$ and a $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$-covering $S \to \mathcal{O}$: the pair of these coverings is called the *Fricke diagram* (cf. [10])

For an element $r \in \boldsymbol{Q} \cup \{1/0\}$, let ℓ_r be a line in $\boldsymbol{R}^2 - \boldsymbol{Z}^2$ of slope r. Then ℓ_r projects to simple loops in T, S, and $\mathcal{O}$, which are *essential*, i.e., each of them does not bound a disk, a disk with one cone point, nor a punctured disk. Conversely, any essential loop α in T, S, or $\mathcal{O}$ is isotopic to the one obtained in this way from a unique $r \in \boldsymbol{Q} \cup \{1/0\}$. Then r is called the *slope* of α, and is denoted $s(\alpha)$.

Since T and S are coverings of the orbifold $\mathcal{O}$, the fundamental groups of T and S are regarded as subgroups of the orbifold fundamental group of $\mathcal{O}$. These groups have the following group presentations:

$$\begin{array}{llcl} (1) & \pi_1(T) & = & < A, B >, \\ (2) & \pi_1(S) & = & < K_0, K_1, K_2, K_3 | K_0K_1K_2K_3 = 1 >, \\ (3) & \pi_1(\mathcal{O}) & = & < P, Q, R | P^2 = Q^2 = R^2 = 1 >, \end{array}$$

Here the generators satisfies the following condition: Put $K = (PQR)^{-1}$, then K is represented by the puncture of $\mathcal{O}$, and we have $K^2 = [A, B]$, $A = KP = RQ$, $B = K^{-1}R = PQ$, $K_0 = K$, $K_1 = K^P$, $K_2 = K^Q$, $K_3 = K^R$, where X^Y denotes YXY^{-1}.

Throughout this paper, we reserve the symbol K to denote the element of $\pi_1(\mathcal{O})$ defined in the above.

Definition 1.1. (1) An ordered pair (A, B) of elements in $\pi_1(T)$ is a *generator pair* of $\pi_1(T)$ if they generate $\pi_1(T)$ and satisfies $[A, B] = K^2$. In this case, A and B, respectively are called the *left* and *right* generators, and (A, AB, B) is called a *generator triple*. The slope of an essential loop in T realizing A [resp. B] is called the *slope* of A [resp. B] and is denoted by $s(A)$ [resp. $s(B)$].

(2) An ordered triple (P, Q, R) of elements of $\pi_1(\mathcal{O})$ is called an *elliptic generator triple* if they generate $\pi_1(\mathcal{O})$ and satisfies $P^2 = Q^2 = R^2 = 1$ and $(PQR)^{-1} = K$. A member of an elliptic generator triple is called an *elliptic generator.*

Proposition 1.2. *(1) There is a one-to-one correspondence between the set of elliptic generator triples and the braid group B_3.*

(2) For any elliptic generator triple (P, Q, R), the following holds:

(2.1) Any three consecutive elements in the following bi-infinite sequence is also an elliptic generator triple.

$$\cdots, P^{K^{-1}}, Q^{K^{-1}}, R^{K^{-1}}, P, Q, R, P^K, Q^K, R^K, \cdots$$

(2.2) (P, R, Q^R) is also an elliptic generator triple.

(3) Conversely, any elliptic generator triple is obtained from (P, Q, R) by successively applying the operations in (2).

(4) If (P, Q, R) is an elliptic generator triple of $\pi_1(\mathcal{O})$, then $(KP, KQ, K^{-1}R)$ is a generator triple of $\pi_1(T)$. Conversely, every generator triple of $\pi_1(T)$ is so obtained.

For each elliptic generator P of $\pi_1(\mathcal{O})$, KP and $K^{-1}P = PK$, respectively, are left and right generators of $\pi_1(T)$ by Proposition 1.2. Further, we see $s(K^{-1}P) = s(PK)$. We define the *slope* $s(P)$ of P by $s(P) = s(K^{-1}P) = s(PK)$. *Throughout this paper, We assume that the slopes of A and B in the group presentation (1) are $1/0$ and $0/1$, respectively and that the slopes of P, Q and R in the group presentation (3) are $1/0$, $1/1$ and $0/1$, respectively.*

The *modular diagram $\mathcal{D}$* is the ideal triangulation of the hyperbolic plane $\boldsymbol{H}^2$ with ideal vertices $\boldsymbol{Q} \cup \{1/0\}$, such that a typical ideal simplex of $\mathcal{D}$ is spanned by $\{\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_1+p_2}{q_1+q_2}\}$ where $\begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} = \pm 1$. The symbol $< s_1, s_2, s_3 >$ denotes the ideal simplex spanned by $\{s_1, s_2, s_3\}$. It also represents the oriented one, where the orientation is given by the ordered triple (s_1, s_2, s_3). Any ideal simplex of $\mathcal{D}$ is the image of the ideal simplex $< 1/0, 1/1, 0/1 >$ by an element of $\mathrm{SL}(2, \boldsymbol{Z})$. We say that an oriented ideal simplex is *positive* if the orientation is coherent with that of $< 1/0, 1/1, 0/1 >$. For each ideal simplex σ in $\mathcal{D}$, the union of the lines in $\boldsymbol{R}^2$ intersecting $\boldsymbol{Z}^2$ with slopes the ideal vertices of σ determines an ideal triangulation of $\boldsymbol{R}^2 - \boldsymbol{Z}^2$ which projects to *maximal arc systems* of T, S, and $\mathcal{O}$.

The (abstract) simplicial complex whose combinatorial structure is equal to that of $\mathcal{D}$ is also denoted by the same symbol $\mathcal{D}$.

Proposition 1.3. *(1) For two elliptic generators P and P', $s(P) = s(P')$ if and only if $P' = P^{K^n}$ for some integer n.*

(2) For any elliptic generator triple (P, Q, R), $< s(P), s(Q), s(R) >$ is a positive oriented 2-simplex of $\mathcal{D}$.

(3) The slopes of two elliptic generator triples span the same 2-simplex of $\mathcal{D}$ if and only if they are related by the operation (2.1) of Proposition 1.2.

(4) For any elliptic generator triple (P, Q, R), $s(Q^R) = s(Q^P)$ holds and it is the image of $s(Q)$ by the reflection in the edge $< s(P), s(R) >$.

(5) Let (A, AB, B) be a generator triple of $\pi_1(T)$. Then (AB^{-1}, A, B) is also a generator triple, and both $< s(A), s(AB), s(B) >$ and $< s(AB^{-1}), s(A), s(B) >$ are positive oriented simplices of $\mathcal{D}$. In particular, $s(AB^{-1})$ is the image of $s(AB)$ by the reflection in the edge $< s(A), s(B) >$.

For a 2-simplex σ of $\mathcal{D}$, the bi-infinite sequence $\cdots, P^{K^{-1}}, Q^{K^{-1}}, R^{K^{-1}}, P, Q, R, P^K, Q^K, R^K, \cdots$ of elliptic generators whose slopes are the vertices of σ is called the *sequence of elliptic generators* associated with σ.

2. Markoff maps and representation spaces

By a *Markoff triple* we mean an ordered triple (x, y, z) of complex numbers satisfying the Markoff equation:

$$x^2 + y^2 + z^2 = xyz.$$

The triple $(0, 0, 0)$ is called the *trivial* Markoff triple. If (x, y, z) is a Markoff map, then (1) we can obviously obtain other Markoff triples by permuting entries, and (2) the triples $(x, y, xy - z)$, $(x, zx - y, z)$, and $(yz - x, y, z)$ are also Markoff triples. On repeating such substitutions, we generate an equivalence class of Markoff triples which has a natural "tree" structure. Such a structure is referred to as a "Markoff map" (see [1]):

Definition 2.1. A *Markoff map* is a map $\phi : \mathcal{D}^{(0)} = \boldsymbol{Q} \cup \{1/0\} \to \boldsymbol{C}$ satisfying the following conditions:

(1) For any 2-simplex $< s_1, s_2, s_3 >$ of $\mathcal{D}$, the triple $(\phi(s_1), \phi(s_2), \phi(s_3))$ is a Markoff triple.

(2) For any pair of 2-simplices $< s_1, s_2, s_3 >$ and $< s_1, s_2, s_4 >$ of $\mathcal{D}$ sharing a common edge $< s_1, s_2 >$, we have

$$\phi(s_3) + \phi(s_4) = \phi(s_1)\phi(s_2).$$

For each Markoff triple (x, y, z) without 0 entries, we have

$$a_1 + a_2 + a_3 = 1, \quad \text{where} \quad a_1 = \frac{x}{yz}, \quad a_2 = \frac{y}{zx}, \quad a_3 = \frac{z}{xy}.$$

We call (a_1, a_2, a_3) a *complex probability.* Conversely, the Markoff triple (x, y, z), up to multiplication by ± 1, is recovered from the complex probability (a_1, a_2, a_3) by the following identities:

$$x^2 = \frac{1}{a_2 a_3}, \quad y^2 = \frac{1}{a_3 a_1}, \quad z^2 = \frac{1}{a_1 a_2}.$$

Explicitly, the transformations $(x, y, z) \to (-x, -y, z)$ and $(x, y, z) \to (x, -y, -z)$ generate a free $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$-action on the space of the nontrivial Markoff triples, and the quotient space is identified with the space $\{(a_1, a_2, a_3) \in \boldsymbol{C}^3 | a_1 + a_2 + a_3 = 1, a_i \neq 0 (1 \leq i \leq 3)\}$ of complex probabilities.

We now introduce the concept of a "complex probability map". Let Σ be a binary tree (a countably infinite simplicial tree, all of whose vertices have degree 3) properly embedded in $\boldsymbol{H}^2$ dual to $\mathcal{D}$. A *directed edge,* $\overrightarrow{e}$, of Σ can be thought of an ordered pair of adjacent vertices of Σ, referred to as the *head* and *tail* of $\overrightarrow{e}$. We introduce the notation $\overrightarrow{e} \leftrightarrow (s_1, s_2; s_3, s_4)$ to mean that s_1, s_2, s_3 and s_4 are the ideal vertices of $\mathcal{D}$ such that (1) $< s_1, s_2 >$ is the dual to $\overrightarrow{e}$ and that (2) $< s_1, s_2, s_3 >$ [resp. $< s_1, s_2, s_4 >$] is dual to the head [resp. tail] of $\overrightarrow{e}$.

Let $\overrightarrow{E}(\Sigma)$ be the set of directed edges of Σ. For a Markoff map ϕ, let $\overrightarrow{E}_\phi(\Sigma)$ be the subset of $\overrightarrow{E}(\Sigma)$ consisting of those directed edges $\overrightarrow{e}$ such that if $\overrightarrow{e} \leftrightarrow (s_1, s_2; s_3, s_4)$ then $\phi(s_1)\phi(s_2) \neq 0$. Then we define a map $\psi : \overrightarrow{E}_\phi(\Sigma) \to \boldsymbol{C}$ by

$$\psi(\overrightarrow{e}) = \frac{\phi(s_3)}{\phi(s_1)\phi(s_2)}.$$

We call ψ the *complex probability map* corresponding to the Markoff map ϕ.

A triple $(\overrightarrow{e}_1, \overrightarrow{e}_2, \overrightarrow{e}_3)$ of elements of $\overrightarrow{E}(\Sigma)$ is said to be *dual* to an oriented simplex $\sigma =< s_1, s_2, s_3 >$ if $\overrightarrow{e}_i$ $(1 \leq i \leq 3)$ is dual to $< s_i, s_{i+1} >$ (where the indices are considered modulo 3) and has σ^*, the vertex of Σ dual to σ, as the head. Then we have the following:

Lemma 2.2. *(1) Let σ be an oriented 2-simplex of $\mathcal{D}$, and $(\overrightarrow{e}_1, \overrightarrow{e}_2, \overrightarrow{e}_3)$ a triple of elements of $\overrightarrow{E}_\phi(\Sigma)$ dual to σ. Then:*

$$\psi(\overrightarrow{e}_1) + \psi(\overrightarrow{e}_2) + \psi(\overrightarrow{e}_3) = 1.$$

We call the triple $(\psi(\overrightarrow{e}_1), \psi(\overrightarrow{e}_2), \psi(\overrightarrow{e}_3))$ the complex probability of ϕ (or the value of ψ) at σ.

(2) For each $\overrightarrow{e} \in \overrightarrow{E}_\phi(\Sigma)$, let $-\overrightarrow{e}$ be the element of $\overrightarrow{E}_\phi(\Sigma)$ obtained from $\overrightarrow{e}$ by reversing the direction. Then

$$\psi(\overrightarrow{e}) + \psi(-\overrightarrow{e}) = 1.$$

(3) Let σ and σ' be positive oriented 2-simplices of $\mathcal{D}$ which are adjacent, and let $(\overrightarrow{e}_1, \overrightarrow{e}_2, \overrightarrow{e}_3)$ and $(\overrightarrow{e}'_1, \overrightarrow{e}'_2, \overrightarrow{e}'_3)$, respectively, be triples of elements of $\overrightarrow{E}_\phi(\Sigma)$ dual to σ and σ', and assume that $e'_1 = -e_3$. Then:

$$\psi(\overrightarrow{e}'_1) = 1 - \psi(\overrightarrow{e}_3), \quad \psi(\overrightarrow{e}'_2) = \frac{\psi(\overrightarrow{e}_2)\psi(\overrightarrow{e}_3)}{1 - \psi(\overrightarrow{e}_3)}, \quad \psi(\overrightarrow{e}'_3) = \frac{\psi(\overrightarrow{e}_1)\psi(\overrightarrow{e}_3)}{1 - \psi(\overrightarrow{e}_3)}.$$

Let Φ be the space of the non-trivial Markoff maps and Ψ that of the complex probability maps. (A Markoff map is called *trivial* if its image is $\{0\}$.) Then we have a natural four to one map $\Phi \to \Psi$. In fact the free $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$-action on the non-trivial Markoff triples induces that on Φ with quotient Ψ.

3. Representation spaces

$\boldsymbol{H}^3$ denotes the upper half-space model of the 3-dimensional hyperbolic space, and its ideal boundary is identified with $\boldsymbol{C} \cup \{\infty\}$. For a Möbius transformation X, the symbol $o(X)$, [resp. $I(X)$, $D(X)$, $E(X)$, $Ih(X)$, $Dh(X)$, $Eh(X)$] denotes the pole [resp. the isometric circle of X, the disk bounded by $I(X)$, $\mathrm{cl}(\boldsymbol{C} - D(X))$, the isometric hemi-sphere of X, the half-ball in $\boldsymbol{H}^3$ bounded by $Ih(X)$, $\mathrm{cl}(\boldsymbol{H}^3 - Dh(X))$]. For a discrete subgroup G of $\mathrm{Isom}(2, \boldsymbol{C})$ with non-trivial stabilizer G_∞, the *extended Ford domain*, denoted by $\tilde{Ph}(G)$, is defined by $\tilde{Ph}(G) = \cap\{Eh(X) | X \in G - G_\infty\}$. A *Ford domain* is the intersection of $\tilde{Ph}(G)$ with a fundamental region of G_∞.

The following lemma can be proved by using the arguments in [2] (Proof of Proposition 1.1).

Lemma 3.1. *(1) Let ρ be a $\mathrm{PSL}(2, \boldsymbol{C})$-representation of $\pi_1(T)$ sending $[A, B] = K^2$ to a parabolic transformation. Then (i) ρ lifts to a $\mathrm{SL}(2, \boldsymbol{C})$-representation $\tilde{\rho}$, and (ii) ρ extends to a representation of $\pi_1(\mathcal{O})$ if and only if $\mathrm{tr}(\tilde{\rho}(K^2)) = -2$.*

(2) Let ρ be a $\mathrm{PSL}(2, \boldsymbol{C})$-representation of $\pi_1(S)$ sending K_i $(0 \le i \le 3)$ to parabolic transformations. Then (i) ρ lifts to a $\mathrm{SL}(2, \boldsymbol{C})$-representation, and (ii) ρ extends to a representation of $\pi_1(\mathcal{O})$ if and only if $\rho(K_i) = \rho(K_j)^{-1}$ whenever $\rho(K_i)$ and $\rho(K_j)$ share the same parabolic fixed point.

A $\mathrm{PSL}(2, \boldsymbol{C})$-representation of $\pi_1(T)$ [resp. $\pi_1(S)$] is said to be *type-preserving* if it satisfies the condition of Lemma 3.1 (1) [resp. (2)]. A $\mathrm{PSL}(2, \boldsymbol{C})$-representation of $\pi_1(\mathcal{O})$ is said to be *type-preserving* if it sends K to a parabolic transformation. For $X = T$, S and $\mathcal{O}$, Let $\mathcal{R}(X)$ be the space of type-preserving $\mathrm{PSL}(2, \boldsymbol{C})$-representations of $\pi_1(X)$ modulo conjugacy, and let $\tilde{\mathcal{R}}(X)$ be the space of $\mathrm{SL}(2, \boldsymbol{C})$-representations modulo conjugacy which project to type-preserving $\mathrm{PSL}(2, \boldsymbol{C})$-representations. Then, by Lemma 3.1, we may identify $\mathcal{R}(T)$, $\mathcal{R}(S)$ and $\mathcal{R}(\mathcal{O})$; they are denoted by the common symbol $\mathcal{R}$. Similarly, we may identify $\tilde{\mathcal{R}}(T)$ and $\tilde{\mathcal{R}}(S)$; they are denoted by the common symbol $\tilde{\mathcal{R}}$. (Note that $\tilde{\mathcal{R}}(\mathcal{O}) = \emptyset$.)

For a $\mathrm{SL}(2, \boldsymbol{C})$-representation $\tilde{\rho} \in \tilde{\mathcal{R}}$, let $\phi_{\tilde{\rho}}$ be a map from $\mathcal{D}^{(0)} = \boldsymbol{Q} \cup \{1/0\}$ to $\boldsymbol{C}$ define by $\phi_{\tilde{\rho}}(r) = \mathrm{tr}(\phi(\alpha_r))$, where α_r is an element of $\pi_1(T)$ represented by the simple loop of slope r. (Note that this definition does not depend on the orientation of the loop nor the choice of the base point of the fundamental group.) Then $\phi_{\tilde{\rho}}$ is a non-trivial Markoff map by the following well-known trace identities for matrices X and Y in $\mathrm{SL}(2, \boldsymbol{C})$:

$$\mathrm{tr}(X)^2 + \mathrm{tr}(Y)^2 + \mathrm{tr}(XY)^2 - \mathrm{tr}(X)\mathrm{tr}(Y)\mathrm{tr}(XY) = 2 + \mathrm{tr}([X, Y]),$$

$$\mathrm{tr}(XY) + \mathrm{tr}(XY^{-1}) = \mathrm{tr}(X)\mathrm{tr}(Y).$$

Further, the correspondence $\tilde{\rho} \longrightarrow \phi_{\tilde{\rho}}$ is a homeomorphism from $\tilde{\mathcal{R}}$ to Φ by [3]. This yields a homeomorphism between $\mathcal{R}$ and Ψ. Therefore we have,

Proposition 3.2. *There are natural homeomorphisms $\tilde{\mathcal{R}} \cong \Phi$ and $\mathcal{R} \cong \Psi$.*

We denote the element of $\mathcal{R}$ corresponding to $\phi \in \Phi$ [resp. $\psi \in \Psi$] by ρ_ϕ [resp. ρ_ψ]. Jorgensen [3] gave a nice construction of the representation ρ_ϕ from a Markoff map ϕ:

Lemma 3.3. *Let ϕ be an element of Φ, put $(x, y, z) = (\phi(1/0), \phi(0/1), \phi(1/1))$, and assume $z \neq 0$. Then the following identities define a representation $\tilde{\rho} \in \tilde{\mathcal{R}}$ corresponding to ϕ:*

$$\tilde{\rho}(A) = \begin{pmatrix} x - y/z & x/z^2 \\ x & y/z \end{pmatrix}, \quad \tilde{\rho}(AB) = \begin{pmatrix} z & -1/z \\ -z & 0 \end{pmatrix}, \quad \tilde{\rho}(B) = \begin{pmatrix} y - x/z & -y/z^2 \\ -y & x/z \end{pmatrix}.$$

Further we have $\tilde{\rho}(K^2) = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix}$. *This representation projects to a representation* $\rho \in \mathcal{R}$ *which satisfies the following identities:*

$$\rho(P) = \begin{bmatrix} y/z & (yz-x)/z^2 \\ -x & -y/z \end{bmatrix}, \quad \rho(Q) = \begin{bmatrix} 0 & -1/z \\ z & 0 \end{bmatrix}, \quad \rho(R) = \begin{bmatrix} -x/z & (xz-y)/z^2 \\ -y & x/z \end{bmatrix},$$

where $[\quad]$ *denotes elements in* $\mathrm{PSL}(2, \boldsymbol{C})$. *In the above, A, B, P, Q, and R are generators in the group presentations (1) and (3) in Section 1.*

The following lemma gives a geometric meaning of the traces:

Lemma 3.4. *Let* ρ *and* (x, y, z) *be as in Lemma 3.3. (However, we do not need to assume* $z \neq 0$.*) Then we have the following:*

$$d_\rho(P, Q) = 2\cosh^{-1}(y/2), \quad d_\rho(Q, R) = 2\cosh^{-1}(x/2), \quad d_\rho(R, P) = 2\cosh^{-1}(z/2).$$

where $d_\rho(P, Q)$, *for example, denotes the complex distance between the axes of the elliptic transformations* $\rho(P)$ *and* $\rho(Q)$.

Corollary 3.5. *Suppose* $z = 0$. *Then* $d_\rho(R, P) = \pm i\pi/2$, *i.e., the axes of the* π*-rotations* $\rho(P)$ *and* $\rho(R)$ *intersect orthogonally. In particular,* $\rho(P)$ *and* $\rho(R)$ *are commuting.*

By this corollary, we see that the image of the $\mathrm{PSL}(2, \boldsymbol{C})$-representation ρ_0 of $\pi_1(\mathcal{O})$ corresponding to the trivial Markoff map is Klein's four group.

In the following, we give a geometric construction (using only a pair of compasses and a ruler) of the $\mathrm{PSL}(2, \boldsymbol{C})$-representations given by Proposition 3.2. We start from a complex probability $(a_1, a_2, a_3) \in \boldsymbol{C}^3$, i.e., a triple of complex numbers satisfying $a_1 + a_2 + a_3 = 1$ and $a_i \neq 0$ $(1 \leq i \leq 3)$. By repeatedly drawing the three vectors a_1, a_2, a_3, we obtain an infinite, possibly singular, broken line $\mathcal{L}$ on the complex plane $\boldsymbol{C}$. Explicitly, $\mathcal{L}$ consists of the vertices $\{o_i | i \in \boldsymbol{Z}\}$ and the directed edge $\overrightarrow{o_i o_{i+1}}$ which is equal to $a_{[i]}$, where $[i]$ denotes the unique integer between 1 and 3 such that $[i] \equiv i \pmod 3$. For each vertex o of $\mathcal{L}$, let P_o be the elliptic transformation of $\boldsymbol{H}^3$ of order 2 whose axis has end-points $o \pm \sqrt{-a_i a_j}$, where a_i and a_j are the entries of the complex probability corresponding to the pair of edges of $\mathcal{L}$ meeting at o. To give a geometric construction of P_o, let C_o be the hemisphere in $\boldsymbol{H}^3$ with center o and with radius $\sqrt{|a_i a_j|}$, the multiplicative mean of $|a_i|$ and $|a_j|$, and let $\boldsymbol{H}_o$ be the hyperbolic plane in $\boldsymbol{H}^3$ whose ideal boundary is the line in $\boldsymbol{C}$ passing through o and bisecting the angle between the pair of edges of $\mathcal{L}$ meeting at o. Then

$$P_o = (\text{reflection in } \boldsymbol{H}_o) \circ (\text{reflection in } C_o).$$

It should be noted that the isometric hemi-sphere $Ih(P_o)$ of P_o is equal to C_o and that the pole of P_o is o. Choose any three consecutive vertices o_1, o_2 and o_3 of $\mathcal{L}$. Then we can see, by an elementary Euclidean geometry, that the product $P_{o_3} P_{o_2} P_{o_1}$ is equal to the parabolic transformation $(z, t) \to (z + 1, t)$ of $\boldsymbol{H}^3$. Hence, we obtain a $\mathrm{PSL}(2, \boldsymbol{C})$-representation ρ of $\pi_1(\mathcal{O})$ belonging to $\mathcal{R}$ by putting

$$\rho(P) = \rho(P_{o_1}), \quad \rho(Q) = \rho(P_{o_2}), \quad \rho(R) = \rho(P_{o_3}).$$

Let ψ be a complex probability map such that $(\psi(\vec{e}_1), \psi(\vec{e}_2), \psi(\vec{e}_3)) = (a_1, a_2, a_3)$, where $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ is dual to the oriented ideal simplex $< 1/0, 1/1, 1/0 >$. Further, we choose o_1, o_2 and o_3 so that

$$\overrightarrow{o_1 o_2} = a_1, \quad \overrightarrow{o_2 o_3} = a_2.$$

Then the representation ρ is equal to that corresponding to $\psi \in \Psi$ by the homeomorphism in Proposition 3.2. If we choose $\mathcal{L}$ so that o_2 is the origin 0 of $\boldsymbol{C}$, then ρ is equal to that given by Lemma 3.3.

Example 3.6. If $(a_1, a_2, a_3) = (1/3, 1/3, 1/3)$, then the image $Im(\rho) = \rho(\pi_1(\mathcal{O}))$ is a Fuchsian group, and its Ford domain is supported by the isometric hemispheres of the elliptic generators of slopes 1/0, 1/1 and 0/1. In fact, we can see that the intersection of a fundamental domain of K and the common exterior of the isometric hemispheres of these elliptic generators satisfy the conditions of the Poincare's theorem on fundamental polyhedra [6].

4. Angle parameters and triangles

Let ρ be an element of $\mathcal{R}$, and let ϕ be a Markoff map such that $\rho = \rho_\phi$. $\mathcal{D}_\rho$ denotes the subcomplex of $\mathcal{D}$ formed by the 2-simplices whose vertices are sent by ϕ to non-zero complex numbers. (Note that this does not depend on the choice of ϕ.) Let $\sigma =< s_1, s_2, s_3 >$ be a positive oriented simplex of $\mathcal{D}_\rho$, and let $\{P_i\}_{i \in \boldsymbol{Z}}$ be the sequence of elliptic generators corresponding to σ such that $s(P_i) = s_{[i]}$. We denote by $\mathcal{L}(\rho; \sigma)$ the infinite possibly singular broken line which is obtained by successively joining the poles $\{o(\rho(P_i))\}$ by edges. By the construction in the previous section, we see

$$\overrightarrow{o(\rho(P_i))o(\rho(P_{i+1}))} = \psi(\overrightarrow{e}_{[i]}),$$

where ψ is the complex probability map corresponding to ρ, and $(\overrightarrow{e}_1, \overrightarrow{e}_2, \overrightarrow{e}_3)$ is dual to σ. Put $a_i = \psi(\overrightarrow{e}_i)$ $(1 \leq i \leq 3)$. Then we have the following:

Lemma 4.1. *The following conditions are equivalent:*

(1) $I(\rho(P_i))$ and $I(\rho(P_{i+1}))$ intersect in two points for some $i \in \boldsymbol{Z}$.

(2) $I(\rho(P_i))$ and $I(\rho(P_{i+1}))$ intersect in two points for any $i \in \boldsymbol{Z}$.

(3) There exists a Euclidean triangle whose edges have lengths $\sqrt{|a_1|}$, $\sqrt{|a_2|}$ and $\sqrt{|a_3|}$.

If the above conditions do not hold, then we have $D(\rho(P_i)) \subset D(\rho(P_{i-1})) \cap D(\rho(P_{i+1}))$ for some i.

Suppose the conditions of the above lemma are satisfied. Let β_i be the angle of the triangle in Lemma 4.1 (3) between the edges of lengths $\sqrt{|a_{i-1}|}$ and $\sqrt{|a_i|}$, where the indices are considered modulo 3.

Suppose further that $\mathcal{L}(\rho; \sigma)$ is simple. Then it divides $\boldsymbol{C}$ into the "upper and lower" parts. For $\epsilon = \pm$, let α_i^ϵ be the angle $(0 \leq \alpha_i^\epsilon < \pi)$ at the vertex $o(\rho(P_i))$. Put $\theta_i^\epsilon = \frac{1}{2}(\alpha_i^\epsilon - 2\beta_i)$. Then it is the signed angle of the arc in $I(\rho(P_i))$ bounded by the fixed point of $\rho(P_i)$ and the point of $I(\rho(P_i)) \cap I(\rho(P_{i\pm1}))$ on the ϵ-side of $\mathcal{L}(\rho; \sigma)$ in $\boldsymbol{C}$. It is determined by $\rho \in \mathcal{R}$, $s_{[i]} \in \mathcal{D}^{(0)}$ and σ. So, we denote it by $\theta_\rho^\epsilon(s_{[i]}, \sigma)$, and call it the *angle parameter* of ρ at $s_{[i]}$ with respect to σ.

Lemma 4.2. *The following identity holds:*

$$\theta_\rho^\epsilon(s_1, \sigma) + \theta_\rho^\epsilon(s_2, \sigma) + \theta_\rho^\epsilon(s_3, \sigma) = \frac{\pi}{2}.$$

Lemma 4.3. *$D(\rho(P_i)) \subset D(\rho(P_{i-1})) \cup D(\rho(P_{i+1}))$ if and only if $\theta_\rho^\epsilon(s_{[i]}, \sigma) \leq 0$ for $\epsilon = \pm$.*

Let σ' be a positive oriented 2-simplex of $\mathcal{D}$ such that $\sigma \cap \sigma' =< s_1, s_3 >$, and assume that σ' also belongs to $\mathcal{D}_\rho$. Let $\{P_i'\}_{i \in \boldsymbol{Z}}$ be the sequence of elliptic generators corresponding to σ'. Then, by Proposition 1.2, we may assume

$$P_1' = P_1, \quad P_2' = P_3, \quad P_3' = P_2^{P_3}.$$

Then by Lemma 2.2 (3), the Euclidean triangle $< o(\rho(P_1)), o(\rho(P_2)), o(\rho(P_3)) >$ is similar to the Euclidean triangle $< o(\rho(P_2')), o(\rho(P_3')), o(\rho(P_4')) >$. In particular, the union of two infinite broken lines $\mathcal{L}(\rho;\sigma)$ and $\mathcal{L}(\rho;\sigma')$ forms a bi-infinite sequence of mutually similar triangles. (The triangles are possibly degenerate and the interiors of the triangles possibly intersect.)

More generally, let $\sigma_1, \sigma_2, \cdots, \sigma_n$ be a finite sequence of 2-simplices of $\mathcal{D}_\rho$, whose duals form the vertex set of a simple path in Σ. Let $\Delta(\rho;\sigma_1,\sigma_2,\cdots,\sigma_n)$ be the union of the bi-infinite broken lines $\{\mathcal{L}(\rho;\sigma_i)\}_{1\le i\le n}$ in $\boldsymbol{C}$. Then this gives a finite array of bi-infinite sequences of mutually similar triangles. In some cases, this gives a triangulation of a region of $\boldsymbol{C}$ (see. Figure 1).

5. Jorgensen's theorem on quasi-Fuchsian groups representing pairs of punctured tori

Let $\mathcal{T}$ be the subspace of $\mathcal{R}$ consisting of the quasi-conformal deformations of the representation in Example 3.6, and let $\partial_{\boldsymbol{Q}}\mathcal{T}$ be the subspace of $\partial\mathcal{T}$ consisting of geometrically finite representations. Let $|\mathcal{D}|$ be the topological realization of abstract simplicial complex $\mathcal{D}$, and put $\mathrm{int}(|\mathcal{D}|) = |\mathcal{D}| - \mathcal{D}^{(0)}$ and $\delta(\mathcal{D}^{(0)}) = \{(v,v) \in |\mathcal{D}| \times |\mathcal{D}| | v \in \mathcal{D}^{(0)}\}$. Then the following is our interpretation of a result of Jorgensen in [3].

Theorem 5.1. *There is a bijection $\nu = \nu^+ \times \nu^-$ from $\mathcal{T} \cup \partial_{\boldsymbol{Q}}\mathcal{T}$ to $|\mathcal{D}| \times |\mathcal{D}| - \delta(\mathcal{D}^{(0)})$ sending $\mathcal{T}$ to $\mathrm{int}(|\mathcal{D}|) \times \mathrm{int}(|\mathcal{D}|)$ which satisfies the following conditions. For each $\rho \in \mathcal{T} \cup \partial_{\boldsymbol{Q}}\mathcal{T}$ let $\ell(\rho)$ be a "straight" line segment joining $\nu^+(\rho)$ and $\nu^-(\rho)$, and let $\sigma^+(\rho) = \sigma_1, \sigma_2, \cdots, \sigma_n = \sigma^-(\rho)$ be the sequence of 2-simplices of $\mathcal{D}$ covering $\ell(\rho)$ in this order, where $\nu^\epsilon(\rho) \in \sigma^\epsilon(\rho)$ $(\epsilon = \pm)$. Then the following holds:*

(1) The configuration $\Delta(\rho;\sigma_1,\sigma_2,\cdots,\sigma_n)$ is non-singular and "dual" to the extended Ford domain $\tilde{Ph}(Im(\rho))$ (see Figure 2). In particular, the extended Ford domain is supported by the isometric hemi-spheres of the images of the elliptic generators each of whose slope is a vertex of some σ_i $(1 \le i \le n)$.

(2) Let s_i^ϵ $(1 \le i \le 3)$ be the vertices of σ^ϵ. Then the angle parameter of ρ at s_i^ϵ with respect to σ^ϵ is non-negative $(1 \le i \le 3)$, and the triple of these angle parameters gives the barycentric coordinate of $\nu^\epsilon(\rho)$ in σ^ϵ.

(3) The restriction of ν to $\mathcal{T}$ is a homeomorphism to $\mathrm{int}(|\mathcal{D}|) \times \mathrm{int}(|\mathcal{D}|)$. In particular, the cellular structure $\mathcal{D} \times \mathcal{D}$ gives that of $\mathcal{T}$.

However, we have not succeeded in proving the whole of the theorem. Though we have proved the part characterizing the combinatorial structures of the Ford domains of the groups in $\mathcal{T}$, we have not proved that part for the groups in $\partial_{\boldsymbol{Q}}(\mathcal{T})$ nor proved that ν is bijective.

6. Jorgensen type theorem for the Riley slice of Schottky groups

Let G_ω be the subgroup of $\mathrm{PSL}(2,\boldsymbol{C})$ generated by the following pair of parabolic transformations:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ \omega & 1 \end{bmatrix}.$$

Following Keen-Series [4], define $\mathcal{S}$ by:

$$\mathcal{S} = \{\rho \in \boldsymbol{C} | \Omega(G_\rho)/G_\rho \text{ is a homeomorphic to a four times punctured sphere } S\}.$$

This has been called the *Riley slice of Schottky groups.* The space of the boundary points of $\mathcal{S}$ which are geometrically finite is denoted $\partial_{\boldsymbol{Q}}(\mathcal{S})$.

We first show that G_ω is represented by a Markoff map with 0. Let $\alpha_{1/0}$ [resp. $\tilde{\alpha}_{1/0}$] be an element of $\pi_1(\mathcal{O})$ [resp. $\pi_1(S)$] represented by a simple loop of slope $1/0$. Let P, Q and R be as in the group presentation (3) of Section 1. Then we may assume:

$$\tilde{\alpha}_{1/0} = \alpha_{1/0}^2 = (QR)^2 \quad \text{in } \pi_1(\mathcal{O}).$$

Let G [resp. $\tilde{G}$] be the quotient group of $\pi_1(S)$ [resp. $\pi_1(\mathcal{O})$] by the normal subgroup normally generated by $\tilde{\alpha}_{1/0} = \alpha_{1/0}^2 = (QR)^2$. Then:

$$G =< K_0 > * < K_3 >, \quad \tilde{G} =< P | P^2 = 1 > * < Q, R | Q^2 = R^2 = (QR)^2 = 1 > .$$

Thus a representation $\rho \in \mathcal{R}$ induces a representations of G and $\tilde{G}$, if and only if $\rho(\tilde{\alpha}_{1/0})$ has order 2: this is equivalent to the condition $\phi(1/0) = 0$, where ϕ is a Markoff map that induces ρ. Let $\mathcal{R}_0$ be the subspace of $\mathcal{R}$ consisting of those representations $\rho \in \mathcal{R}$ satisfying these (mutually equivalent) conditions. Let $\Phi_0 = \{\phi \in \Phi | \phi(1/0) = 0\}$, and let $(\overrightarrow{e}_1, \overrightarrow{e}_2, \overrightarrow{e}_3)$ be the triple of elements of $\overrightarrow{E}(\Sigma)$ dual to the oriented 2-simplex $< 0/1, 1/2, 1/1 >$. For $\phi \in \Phi_0$, put $x = \phi(0/1)$. Then $\phi(1/1) = \pm ix$ and $\phi(1/2) = \pm ix^2$. Hence, the complex probability map ψ corresponding to ϕ satisfies the identity:

$$(\psi(\overrightarrow{e}_1), \psi(\overrightarrow{e}_2), \psi(\overrightarrow{e}_3)) = (a, -a, 1), \quad \text{where } a = 1/x^2.$$

Let Ψ_0 be the subspace of Ψ consisting of those elements satisfying the above condition for some $a \in \boldsymbol{C}^*$. Then we can identify Ψ_0 with $\boldsymbol{C}^*$ by the correspondence $\psi \mapsto a$. The $\mathrm{PSL}(2, \boldsymbol{C})$-representation ρ corresponding to $\psi \in \Psi_0$ satisfies:

$$\rho(P) = \begin{bmatrix} -i & -i \\ 0 & i \end{bmatrix}, \quad \rho(K_0) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \rho(K_3) = \begin{bmatrix} 1 & 0 \\ \omega & 1 \end{bmatrix}, \quad \text{where } \omega = 1/a.$$

In particular $\rho(P)$ acts on $\boldsymbol{C}$ as the π-rotation about the point $1/2$, and we have $\rho(G) = G_\omega$. Hence the Riley slice $\mathcal{S}$ of schottky groups is regarded as a region in $\Psi_0 \cong \boldsymbol{C}^*$.

To study the Ford domain of the groups in $\mathcal{S}$, we prepare some concepts and lemmas. By the term an *elliptic generator* [resp. an *elliptic generator triple*, a *sequence of elliptic generators*] of $\tilde{G}$, we mean the image of that of $\pi_1(\mathcal{O})$. Let Λ be the subgroup of the automorphisms of the simplicial complex $\mathcal{D}$ generated by the reflection in the edge $< 1/0, 0/1 >$ and that in the edge $< 1/0, 1/1 >$. Then Λ is isomorphic to the infinite dihedral group, and the region bounded by the edges $< 1/0, 0/1 >$ and $< 1/0, 1/1 >$ is its fundamental region: in fact, it is identified with the quotient $\mathcal{D}/\Lambda$. The quotient $\mathcal{D}^{(0)}/\Lambda = \boldsymbol{Q} \cup \{1/0\}/\Lambda$ is identified with $\boldsymbol{Q} \cap [0, 1]$. By Theorem 1.1 of [5], two (left or right) generators of $\pi_1(T)$ determine the same element in $\tilde{G}$ if and only if their slopes are equal in $\boldsymbol{Q} \cup \{1/0\}/\Lambda$. Hence we can define the *slope* $s(Q)$ of an elliptic generator Q of $\tilde{G}$ as the image in $\boldsymbol{Q} \cup \{1/0\}/\Lambda$ of the slope of an elliptic generator of $\pi_1(\mathcal{O})$ which projects to Q. Then the following holds:

Lemma 6.1. *Suppose two elliptic generators Q and Q' of $\tilde{G}$ have the same slopes. Then Q' is equal to Q or Q^P modulo conjugation by an element of $< K >$.*

The following example is the starting point of our investigation of the Riley slice.

Example 6.2. The region $0 < |a| \leq 1/4$ (equivalently, the region $\omega \geq 4$) is contained in the Riley slice. To see this, let $\{P_i\}$ be the sequence of elliptic generators of $\tilde{G}$ corresponding to the 2-simplex $< 1/1, 1/2, 0/1 >$, such that $s(P_i)$ is equal to $1/1$, $1/2$ or $0/1$ according as i is congruent to 1, 2 or 3 modulo 3. The for any $\rho \in \mathcal{R}_0$ and for any integer

n we have $I(\rho(P_{1+3n})) = I(\rho(P_{3+3n}))$. Suppose the parameter a satisfies the condition $0 < |a| \leq 1/4$, then we can see $\{I(\rho(P_{1+3n}))\}_{n\in\boldsymbol{Z}}$ are disjoint and that its exterior is the extended Ford domain of $Im(\rho)$.

Let $\sigma_1, \sigma_2, \cdots, \sigma_n$ be a finite sequence of mutually distinct 2-simplices of $\mathcal{D}/\Lambda$, such that $\sigma_1 =< 1/0, 1/1, 0/1 >$ and that σ_i and σ_{i+1} are adjacent for each i $(1 \leq i \leq n-1)$. For $\rho \in \mathcal{R}_0$, let $\Delta_0(\rho; \sigma_1, \sigma_2, \cdots, \sigma_n)$ be the union of the broken lines, $\{\mathcal{L}(\rho;\sigma_i)\}_{2\leq i\leq n}$, and let $\tilde{\Delta}_0(\rho; \sigma_1, \sigma_2, \cdots, \sigma_n)$ be the union of $\Delta_0(\rho; \sigma_1, \sigma_2, \cdots, \sigma_n)$ and its image by $\rho(P)$, the π-rotation about $1/2$. Then we conjecture that the following analogue of Jorgensen's theorem holds:

Perhaps Theorem 6.3. *Let $\mathcal{D}_0 = (\mathcal{D} - \{1/0\})/\Lambda$. Then there is a map from $\mathcal{S} \cup \partial_{\boldsymbol{Q}}(\mathcal{S})$ onto $\mathcal{D}_0$ which satisfies the following conditions. For each $\rho \in \mathcal{S} \cup \partial_{\boldsymbol{Q}}\mathcal{S}$, let $\ell(\rho)$ be a "straight" line segment joining $1/0$ and $\nu(\rho)$. Let $\sigma_1 =< 1/0, 1/1, 0/1 >, \sigma_2, \cdots, \sigma_n$ be the sequence of 2-simplices of $\mathcal{D}$ covering $\ell(\rho)$ in this order. Then the following holds:*

(1) The configuration $\tilde{\Delta}_0(\rho; \sigma_1, \sigma_2, \cdots, \sigma_n)$ is non-singular and "dual" to the extended Ford domain of $\tilde{Ph}(Im(\rho))$. In particular, the extended Ford domain is supported by the isometric hemi-spheres of the images of the elliptic generators each of whose slope is a vertex of some σ_i $(2 \leq i \leq n)$.

(2) Suppose $n \geq 2$, and let s_i $(1 \leq i \leq 3)$ be the vertices of σ_n. Then the angle parameter of ρ at s_i with respect to σ_n is non-negative $(1 \leq i \leq 3)$, and the triple of these angle parameters gives the barycentric coordinate of $\nu(\rho)$ in σ_n.

(3) The restriction of ν to $\mathcal{S}$ determines a homeomorphism from $\mathcal{S}/\sim$ to $\mathcal{D}_0$, where $\sim$ is the equivalence relation defined by $\omega \sim -\bar{\omega}$. In particular, the cellular structure of $\mathcal{D}_0$ induces that of $\mathcal{S}$.

7. Continuous family of cone manifolds approaching to 2-bridge knot complements

A *trivial tangle* is a pair (B^3, t), where B^3 is a 3-ball and t is a union of two arcs in B^3 which is parallel to a union of two mutually disjoint arcs in ∂B^3. A *meridian* of (B^3, t) is a simple loop on $\partial B^3 - t$ which bounds a disk in B^3 separating the components of t. A *rational tangle* is a trivial tangle (B^3, t) equipped with a homeomorphism from $\partial(B^3, t)$ to $(\boldsymbol{R}^2, \boldsymbol{Z}^2)/\Gamma$. The *slope* of a rational tangle is defined to be the slope of its meridian. The rational tangle of slope p/q is denoted $(B^3, t(p/q))$. Then we can identify $\pi_1(B^3 - t(p/q))$ with $\pi_1(S)/ < \tilde{\alpha}_{p/q} >$. The $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$-symmetry of S extends to that of $(B^3, t(p/q))$, and the orbifold fundamental group $\pi_1((B^3 - t(p/q))/(\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2))$ is identified with $\pi_1(\mathcal{O})/ < \alpha^2_{p/q} >$. The *2-bridge knot of slope p/q*, denoted $K(p/q)$, is the "sum" of the rational tangles of slopes $1/0$ and p/q. Then the knot group $G(p/q) = \pi_1(S^3 - K(p/q))$ is identified with $\pi_1(S)/ < \tilde{\alpha}_{1/0}, \tilde{\alpha}_{p/q} >$. The $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$-symmetry of S extends to that of $(S^3, K(p/q))$, and the orbifold fundamental group $\tilde{G}(p/q) = \pi_1((S^3 - K(p/q))/(\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2))$ is identified with $\pi_1(\mathcal{O})/ < \alpha^2_{1/0}, \alpha^2_{p/q} >$. These observations implies that the non-elementary parabolic $SL(2, \boldsymbol{C})$-representations of $G(p/q)$ correspond to the Markoff maps which take the value 0 at $1/0$ and p/q.

Following Riley [8], we define the *Heckoid group* $G(p/q; n)$ $(n \geq 1)$ and the *extended Heckoid group* $\tilde{G}(p/q; k)$ $(k \geq 2)$ by:

$$G(p/q; n) = \pi_1(S)/ < \tilde{\alpha}_{1/0}, \tilde{\alpha}^n_{p/q} >, \quad \tilde{G}(p/q; k) = \pi_1(\mathcal{O})/ < \alpha^2_{1/0}, \alpha^k_{p/q} > .$$

It should be noted that the extended Heckoid group $\tilde{G}(0/1; k)$ $(k \geq 3)$ is isomorphic to the *classical Hecke group* $H(2, q) =< x, y | x^2 = y^k = 1 >$.

Let $K(p/q;n)$ be the orbifold with underlying space S^3-K and with the cone type singularity of angle $2\pi/n$ along the "lower tunnel" τ (i.e., an unknotted arc joining the two components of $t(p/q)$ in B^3 and intersecting the meridian disk transversely in one point). Then the orbifold fundamental group of $K(p/q;n)$ is isomorphic to $G(p/q;n)$. This orbifold can be considered as a cone manifold, denoted $C(p/q;\theta)$, with cone angle $\theta=2\pi/n$. The $\boldsymbol{Z}_2\oplus\boldsymbol{Z}_2$-symmetry of $(S^3,K(p/q))$ determines that of the cone manifold $C(p/q;\theta)$, and its quotient cone manifold is denoted $\mathcal{O}(p/q;2\theta)$. If $\theta=\pi/n$ for some positive integer n, then the cone manifold is an orbifold with orbifold fundamental group $\tilde{G}(p/q;2n)$. If $\theta=0$, then the (orbifold) fundamental groups of $C(p/q;\theta)$ and $\mathcal{O}(p/q;2\theta)$ are regarded as the rational boundary points of the Riley slice $\mathcal{S}$ of slope p/q (i.e., $\alpha_{p/q}$ is an accidental parabolic transformation).

Conjecture 7.1. *(1) Suppose* $0\le\theta<\pi$. *Then* $\mathcal{O}(p/q;2\theta)$ *is a hyperbolic cone manifold. Let* ρ_θ *be its holonomy. It induces a* $\mathrm{PSL}(2,\boldsymbol{C})$*-representation of* $\pi_1(\mathcal{O})$ *which belongs to* $\mathcal{R}_0$*: we denote it by the same symbol* ρ_θ*. Let* $\sigma_1,\sigma_2,\cdots,\sigma_n$ *be the sequence of 2-simplices of* $\mathcal{D}/\Lambda$ *joining the vertices* $1/0$ *and* p/q*. Then the configuration* $\tilde{\Delta}_0(\rho_\theta;\sigma_1,\sigma_2,\cdots,\sigma_n)$ *is non-singular, and the "extended Ford domain" of* $\mathcal{O}(p/q;2\theta)$ *is "dual" to* $\tilde{\Delta}_0(\rho_\theta;\sigma_1,\sigma_2,\cdots,\sigma_n)$*. Here the extended Ford domain of* $\mathcal{O}(p/q;2\theta)$ *means a fundamental region supported by isometric hemi-spheres of elements of* $\mathrm{Im}(\rho_\theta)$ *mudulo the action of the Euclidean translation* $\rho_\theta(K)=\begin{bmatrix}1&1\\0&1\end{bmatrix}$.

(2) As θ *approaches* π*, the representation* ρ_θ *converges to a representation, denoted* ρ_π*.*

(i) Suppose $p\not\equiv\pm1 \pmod q$*. Then* ρ_π *induces a faithful discrete representation of the fundamental group* $\tilde{G}(p/q)$ *of the orbifold* $\mathcal{O}(p/q;\pi)=(S^3-K(p/q))/(\boldsymbol{Z}_2\oplus\boldsymbol{Z}_2)$*. Let* P *and* Q*, respectively, be elliptic generators of slopes* $1/0$ *and* p/q*. Then* $\rho_\pi(P)$ *and* $\rho_\pi(Q)$ *fix* ∞*, and hence they induce (Euclidean)* π*-rotations of* $\boldsymbol{C}$*. The configuration* $\Delta_0(\rho_\pi;\sigma_1,\sigma_2,\cdots,\sigma_{n-1})$ *is "non-singular", and the its image by the action of the infinite dihedral group* $<\rho_\pi(P),\rho_\pi(Q)>$ *give a triangulation of* $\boldsymbol{C}$*. This triangulation is dual to the Ford domain of* $\mathrm{Im}(\rho_\pi(P))$*.*

(ii) Suppose $p\equiv\pm1 \pmod q$*.* ρ_π *induces a faithful discrete* $\mathrm{PSL}(2,\boldsymbol{R})$*-representation of the quotient of the fundamental group* $\tilde{G}(\pm1/q)$ *by the infinite cyclic normal subgroup. In particular,* $\mathrm{Im}(\rho_\pi)$ *is equal to the classical Hecke group* $G(2,q)$*.*

It is easy to verify the above conjecture for $p/q=0/1$ and $1/2$ (cf. Propositions 3.1 and 3.2 in [7]). Figure 3 shows computer experiments, by using the program written by the third author [11], which convince us that the conjecture is valid for $p/q=2/5$.

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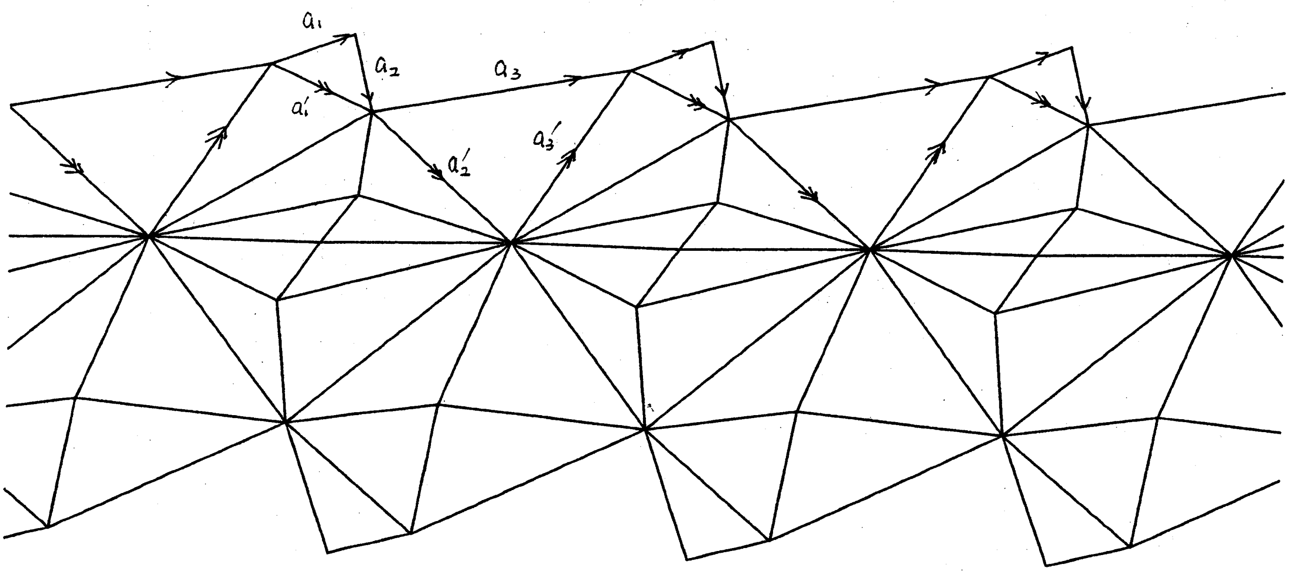


Figure 1

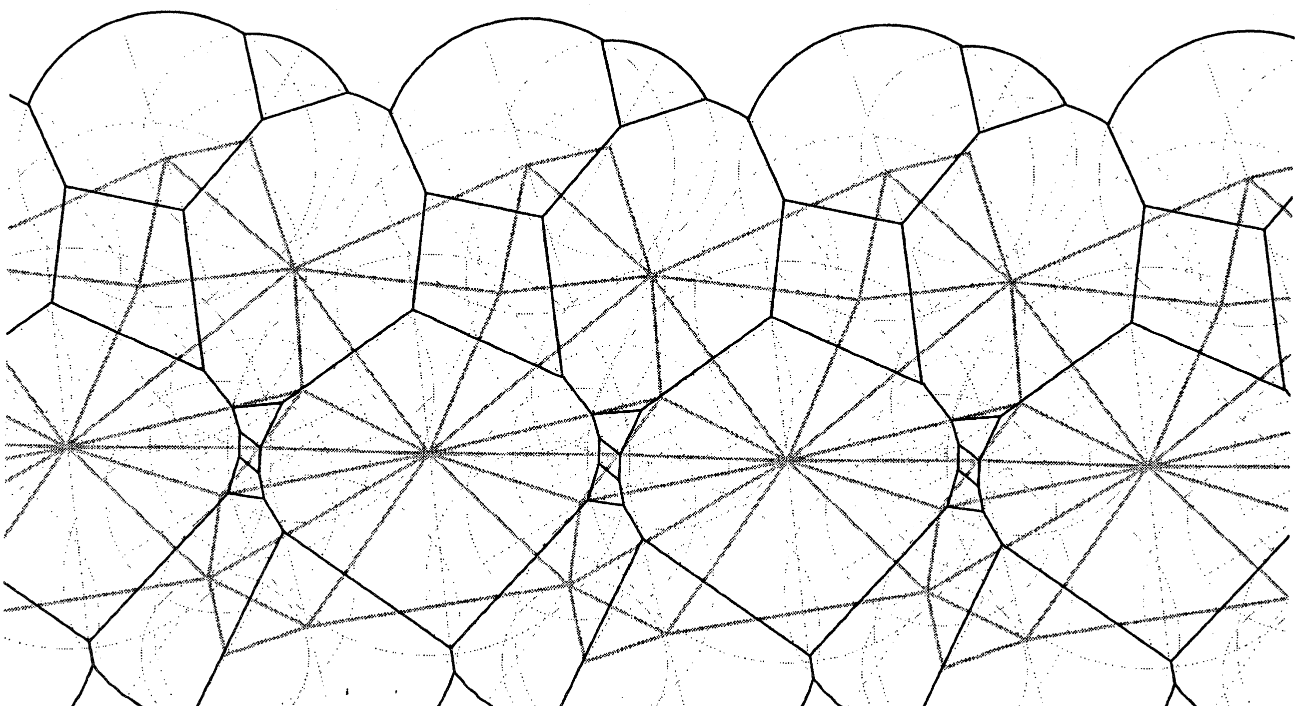
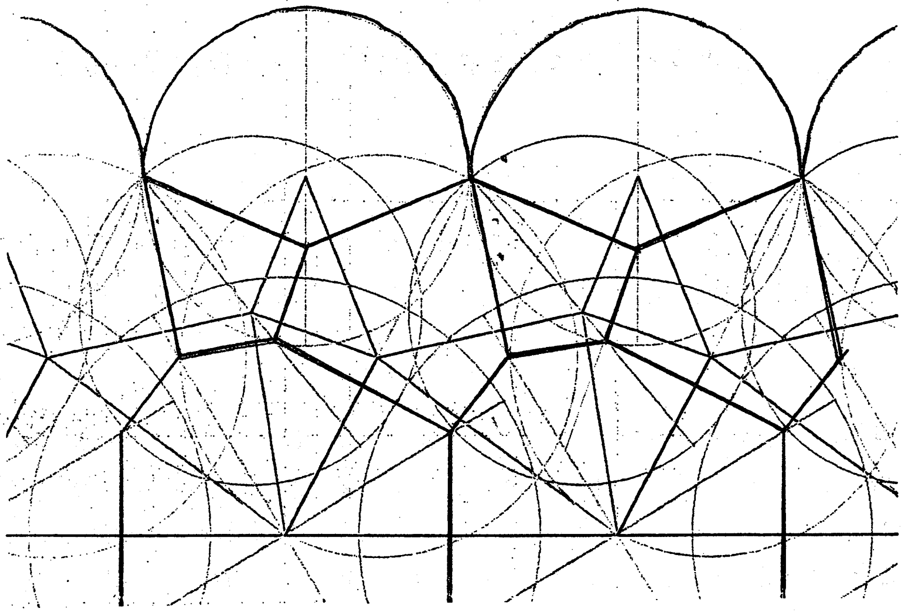
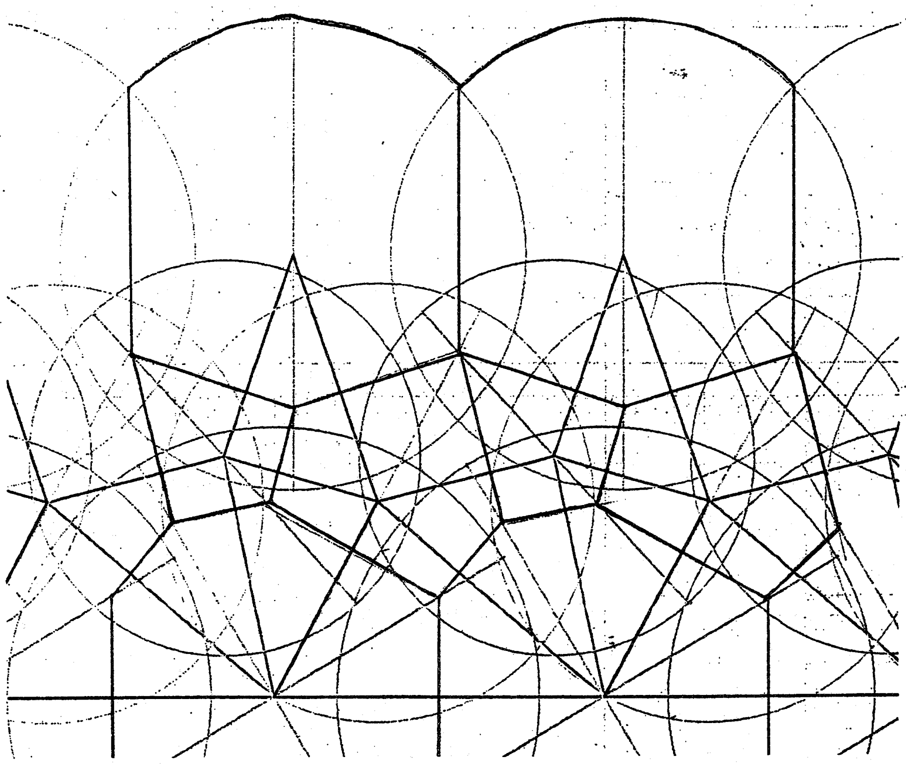


Figure 2



$\theta(2/5, 0)$



$\theta(2/5, \pi/2)$

Figure 3